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COMPARATIVE STATIC PROPOSITIONS IN CONSUMER LOCATION

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The consumer's world is extended to include spatially separate employment and commodity markets. Income available for the purchase of commodities varies with location because of commuting costs. The delivered prices of commodities depend on consumer location. Utility maximization involves the simultaneous choice of the commodity bundle and the optimal location. A generalized Slutsky equation is presented in which the substitution effect involves site substitution as well as commodity substitution by the consumer. Numerous other comparative static results are presented and in general, though the effects are not unwieldy, the signs of the effects are indeterminate.

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1. Introduction

This inquiry was motivated by my desire to investigate the validity of the classical propositions of consumer theory in a model where the consumer's choice of residential location is a variable. For example, is the "law of demand", namely that a consumer normally (Giffen goods ruled out) buys less of commodity i when its price rises, valid when a consumer can shift his site? For the simple (and I suggest orthodox) model developed in this paper, the answer is no; or more specifically, that we cannot be certain of its validity.

In Section 2, the model is developed and in Section 3 a set of propositions in comparative statics are presented. Substitution and income effects are analyzed with the model as well as new effects, denoted transportation cost effects.

2. The Model and Equilibrium

The consumer has a quasi-concave utility function u defined on non-negative quantities of commodities, x_1 and x_2 ¹. The consumer travels from his residence to his

place of work at transportation cost t_3 per unit distance (round trip). His income y is received at his place of work. The consumer travels to markets 1 and 2 for commodities 1 and 2 respectively at transportation costs t_1 and t_2 per unit distance (round trip) respectively.

The economic geography is the line in Figure 2.1

We assume that the consumer is in equilibrium at point z and present the appropriate equilibrium conditions below. The consumer's income y is divided among purchases of commodities 1 and 2 and transportation costs in the following manner:

$$(2.1) \quad y = [p_1 + t_1(z-k_1)]x_1 + [p_2 + t_2(k_2-z)]x_2 + t_3(k_3-z)$$

where p_1 and p_2 are mill prices for commodities 1 and 2 respectively.

The consumer's problem of choice is to select x_1 , x_2 , and z so as to maximize

$$(2.2) \quad u = u(x_1, x_2)$$

subject to (2.1) and non-negativity conditions on x_1 , x_2 , and z . The first order equilibrium conditions to this problem are:

$$(2.3) \quad \frac{\partial u}{\partial x_1} - \lambda \hat{p}_1 = 0$$

$$(2.4) \quad \frac{\partial u}{\partial x_2} - \lambda \hat{p}_2 = 0$$

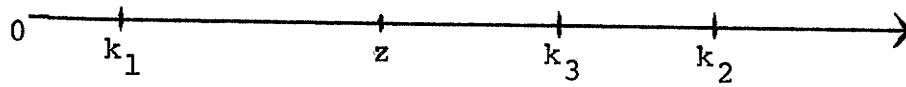


Figure 2.1

$$(2.5) \quad y - \hat{p}_1 x_1 - \hat{p}_2 x_2 - (k_3 - z)t_3 = 0$$

$$(2.6) \quad \lambda \hat{t} = 0 \quad k_1 \leq z \leq k_3$$

where λ is the Lagrangian multiplier

$$\hat{p}_1 = p_1 + (z - k_1)t_1$$

$$\hat{p}_2 = p_2 + (k_2 - z)t_2$$

$$\hat{t} = -t_1 x_1 + t_2 x_2 + t_3$$

Equalities in conditions (2.3) to (2.6) imply that an interior solution exists for the optimizing problem. We assume this to be true in order to simplify the analysis.²

Conditions (2.3) and (2.4) indicate that in equilibrium, the ratio of the marginal utilities of the commodities equals the ratio of their respective delivered prices. Condition (2.5) indicates that in equilibrium all income will be spent. Condition (2.6) indicates that a consumer's expenditures which depend explicitly on his location remain unchanged for a marginal shift in location.

λ is non-zero and hence $\hat{t} = 0$ or $x_1 = \frac{t_2}{t_1} x_2 + \frac{t_3}{t_1}$. The

analogous equilibrium condition occurs in the analysis of residential site choice in a monocentric city, for example Solow(1972)

The consumer's equilibrium is illustrated in Figure 2.2. In Figure 2.2, the budget constraint is tangent to the indifference curve. Hence conditions (2.3) - (2.5) are satisfied. The tangency of the budget constraint and indifference curve occur on line $-\frac{t_3}{t_2}, \frac{t_3}{t_1}$ satisfying condition (2.6). The

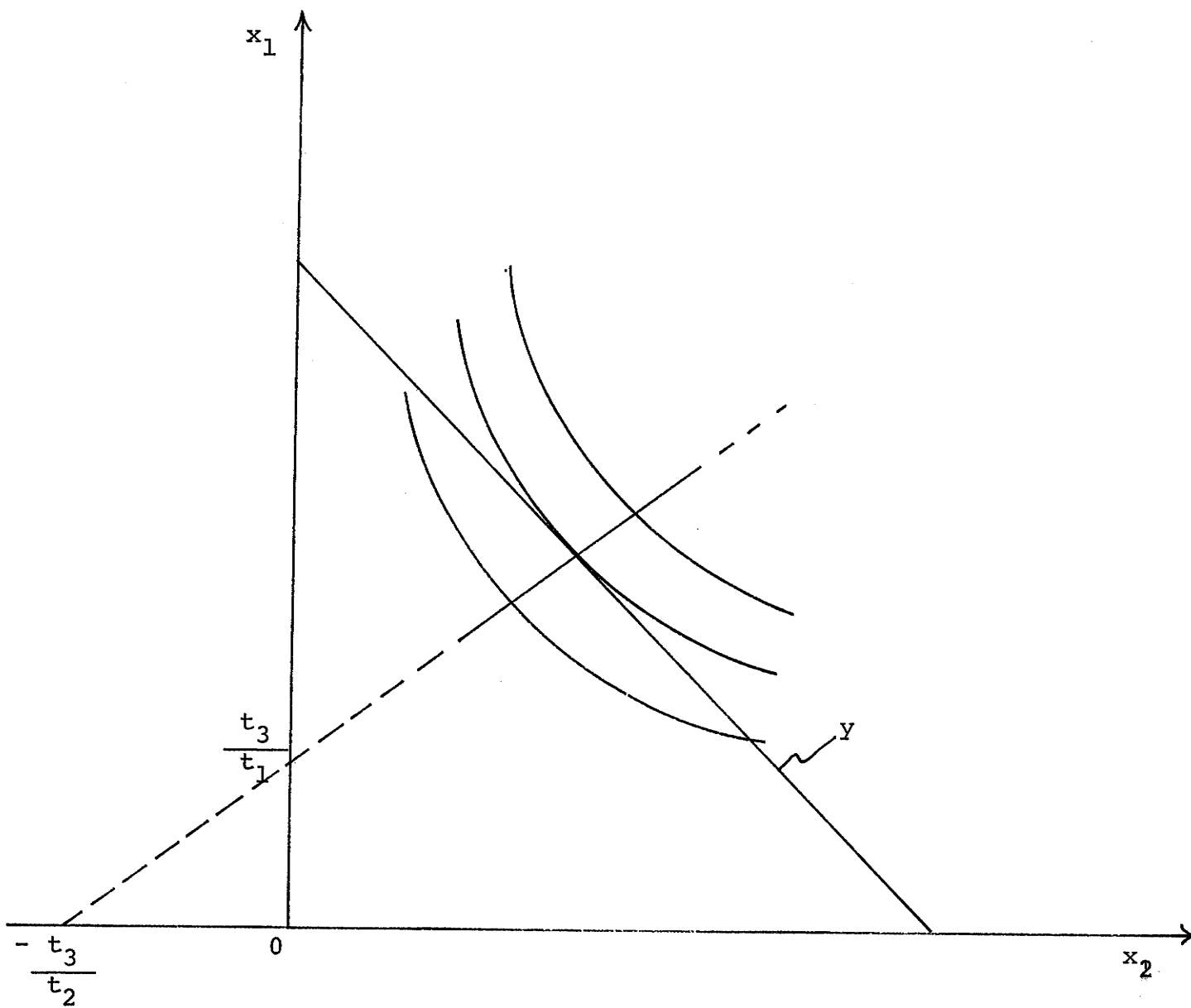


Figure 2.2

slope of the budget constraint is equal to the ratio of the delivered prices \hat{p}_1 and \hat{p}_2 .

The second order equilibrium conditions which must be satisfied are that the last two principal minors of matrix (2.7),

$$(2.7) \quad \begin{bmatrix} 0 & \hat{t} & \lambda t_2 & -\lambda t_1 \\ \hat{t} & 0 & -\hat{p}_2 & -\hat{p}_1 \\ \lambda t_2 & -\hat{p}_2 & u_{22} & u_{21} \\ -\lambda t_1 & -\hat{p}_1 & u_{12} & u_{11} \end{bmatrix}$$

must alternate in sign starting with the first one positive.

3. Analysis of Comparative Statics

We totally differentiate equations (2.3) to (2.6) to obtain the elements in Table 3.1. Note that the signs on the RHS have been changed.

The determinant of the ij^{th} minor, of the 4 by 4 matrix on the LHS of Table 3.1, formed from the matrix by deleting the i^{th} row and the j^{th} column, we indicate by D_{ij} . Using Cramer's rule we can solve for the effect of a change in a parameter on the choice of a control variable (x_1 , x_2 , λ , z) by the consumer. For example, using Cramer's rule,

$$\frac{dx_1}{dp_1} = -\lambda \frac{D_{11}}{D} + x_1 \frac{D_{31}}{D}$$

where D is the determinant of the 4 by 4 matrix on the LHS of Table 3.1.³

dx_1	dx_2	$d\lambda$	dz	dp_1	dp_2	dy	dt_1	dt_2	dt_3	dk_1	dk_2	dk_3
u_{11}	u_{12}	$-\hat{p}_1$	$-\lambda t_1$	λdp_1	0	0	$+\lambda(z-k_1)dt_1$	0	0	$-\lambda t_1 dk_1$	0	0
u_{21}	u_{22}	$-\hat{p}_2$	$-\lambda t_2$	0	$+\lambda dp_2$	0	0	$+\lambda(k_2-z)dt_2$	0	0	$+\lambda t_2 dk_2$	0
$-\hat{p}_1$	$-\hat{p}_2$	0	\hat{t}	$+x_1 dp_1$	$+x_2 dp_2$	$-dy$	$+(z-k_1)x_1 dt_1$	$+(k_2-z)x_2 dt_2$	$+(k_3-z)dt_3$	$-x_1 t_1 dk_1$	$+x_2 t_2 dk_2$	$+t_3 dk_3$
$-\lambda t_1$	$+\lambda t_2$	\hat{t}	0	0	0	0	$+\lambda x_1 dt_1$	$-\lambda x_2 dt_2$	$-\lambda dt_3$	0	0	0

TABLE 3.1

The complete set of comparative static effects are displayed in Table 3.2. On the horizontal dimension we have the label for the impact on the consumer's control variables and along the vertical dimension we have the change in a particular parameter leading to the response by the consumer. The quantitative effect is recorded in the upper part of each cell, assumptions on signs are recorded below and the sign of the terms in the effect are recorded along the bottom of the relevant cell.

Three lines of investigation are suggested by the array of effects in Table 3.2.

i) Signs: We observe that only for income effects, $\frac{dx_1}{dy}$ and $\frac{dx_2}{dy}$, and employment site shift effects $\frac{dx_1}{dk_3}$ and $\frac{dx_2}{dk_3}$ we can make reasonably strong statements about the signs

of the effects. Note that we are free to assign sign values only to variables \hat{t} , u_{12} , and u_{21} . All other variables have signs known a priori, and in particular u_{11} , and u_{22} are assumed to be negative even when not explicitly indicated in Table 3.2.

ii) Linear Combinations of Effects: We observe reading down a column that certain effects are simple combinations of other effects. Note that the income effect

Table 3.2

	dx_1	dx_2	$d\lambda$	dz
dp_1	$\frac{\lambda D_{11}}{D} + x_1 \frac{D_{31}}{D}$ $u_{22} < 0$ $\epsilon < 0$ $D < 0$ $-$	$\frac{\lambda D_{12}}{D} + x_1 \frac{D_{32}}{D}$ $?$ $\epsilon > 0$ $u_{11} < 0$ $u_{21} < 0$ $+$	$\frac{\lambda D_{13}}{D} + x_1 \frac{D_{33}}{D}$ $u_{22} < 0$ $u_{21} < 0$ $\epsilon < 0$ $+$	$\frac{\lambda D_{14}}{D} + x_1 \frac{D_{34}}{D}$ $?$ $?$
dp_2	$\frac{\lambda D_{21}}{D} + x_2 \frac{D_{31}}{D}$ $?$ $u_{22} < 0$ $u_{12} < 0$ $\epsilon < 0$ $+$	$\frac{\lambda D_{22}}{D} + x_2 \frac{D_{32}}{D}$ $\epsilon > 0$ $u_{11} < 0$ $-$ $\epsilon > 0$ $u_{11} < 0$ $u_{21} < 0$ $+$	$\frac{\lambda D_{23}}{D} + x_2 \frac{D_{33}}{D}$ $u_{11} < 0$ $u_{12} < 0$ $\epsilon > 0$ $+$	$\frac{\lambda D_{24}}{D} + x_2 \frac{D_{34}}{D}$ $u_{11} < 0$ $u_{12} > 0$ $\epsilon > 0$ $-$
dy	$-\frac{D_{31}}{D}$ $u_{22} < 0$ $u_{12} < 0$ $\epsilon < 0$ $-$	$-\frac{D_{32}}{D}$ $\epsilon > 0$ $u_{11} < 0$ $u_{21} < 0$ $-$	$-\frac{D_{33}}{D}$ $??$	$-\frac{D_{34}}{D}$ $?$
t_1	$(z-k_1) \frac{dx_1}{dp_1} + \lambda x_1 \frac{D_{41}}{D}$ $u_{22} < 0$ $u_{12} > 0$ $\epsilon < 0$ $+$	$(z-k_1) \frac{dx_2}{dp_1} + \lambda x_1 \frac{D_{42}}{D}$ $u_{11} < 0$ $u_{21} > 0$ $\epsilon > 0$ $-$	$(z-k_1) \frac{d\lambda}{dp_1} + \lambda x_1 \frac{D_{43}}{D}$ $?$	$(z-k_1) \frac{dz}{dp_1} + \lambda x_1 \frac{D_{44}}{D}$ $u_{21} > 0$ $u_{12} > 0$ $-$
t_2	$(k_2-z) \frac{dx_1}{dp_2} - \lambda x_2 \frac{D_{41}}{D}$ $u_{22} < 0$ $u_{12} > 0$ $\epsilon < 0$ $-$	$(k_2-z) \frac{dx_2}{dp_2} + \lambda x_2 \frac{D_{42}}{D}$ $u_{11} < 0$ $u_{21} > 0$ $\epsilon > 0$ $+$	$(k_2-z) \frac{d\lambda}{dp_2} - \lambda x_2 \frac{D_{43}}{D}$ $?$	$(k_2-z) \frac{dz}{dp_2} - \lambda x_2 \frac{D_{44}}{D}$ $u_{21} > 0$ $u_{12} > 0$ $+$

Table 3.2 (cont'd)

	dx_1	dx_2	$d\lambda$	dz
dt_3	$(k_3 - z) \frac{dx_1}{dy} - \lambda x_2 \frac{D_{41}}{D}$ $u_{22} < 0$ $u_{12} > 0$ $\hat{e} < 0$ $-$	$(k_3 - z) \frac{dx_2}{dy} - \lambda \frac{D_{42}}{D}$ $u_{11} < 0$ $u_{21} > 0$ $\hat{e} > 0$ $+$	$(k_3 - z) \frac{d\lambda}{dy} - \lambda \frac{D_{43}}{D}$ $?$	$(k_3 - z) \frac{dz}{dy_2} - \lambda \frac{D_{44}}{D}$ $u_{21} > 0$ $u_{12} > 0$ $+$
dk_1	$-t_1 \frac{dx_1}{dp_1}$ $?$	$-t_1 \frac{dx_2}{dp_1}$ $?$	$-t_1 \frac{d\lambda}{dp_1}$ $?$	$-t_1 \frac{dz}{dp_1}$ $?$
dk_2	$t_2 \frac{dx_1}{dp_2}$ $?$	$t_2 \frac{dx_2}{dp_2}$ $?$	$t_2 \frac{d\lambda}{dp_2}$ $?$	$t_2 \frac{dz}{dp_2}$ $?$
k_3	$-t_3 \frac{dx_1}{dy}$ $u_{22} < 0$ $u_{12} > 0$ $\hat{e} < 0$ $+$	$-t_3 \frac{dx_2}{dy}$ $\hat{e} > 0$ $u_{11} < 0$ $u_{21} < 0$ $+$	$-t_3 \frac{d\lambda}{dy}$ $?$	$-t_3 \frac{dz}{dy}$ $?$

$\frac{dx_1}{dy}$ occurs in the price effects $\frac{dx_1}{dp_1}$ and $\frac{dx_1}{dp_2}$. These are

familiar results. However note in addition that the effects resulting from changing a transportation cost (per unit distance) have the price effects and/or income effects contained in them. The weights multiplying the price and/or income effects are distances between markets (including employment site) and the consumer's residence. Also the shift in location of market effects (e.g. $\frac{dx_1}{dk_1}$) are

multiples of the price effects where the weighting factor is the transportation cost per unit distance. Hence shifting the site of market 1 marginally has the same effect as shifting the mill price of commodity 1 magnified by $-t_1$.

Observe also that, $d\lambda$ and dz , shifts in the marginal utility of income and shifts in residence site respectively, have the same aspects of decomposability as the commodity quantity shifts, dx_1 and dx_2 .

iii) Income, Substitution, and Transportation Cost Effects: Consider the term $\frac{dx_1}{dp_1}$. When the consumer "moves

along" his indifference curve, then $du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0$

and at new prices, $du = p_1 dx_1 + p_2 dx_2 = 0$.

Also when a consumer shifts his consumption of quantities x_1 and x_2 (when for example he "moves along" his indifference curve) and shifts his site so that with the new (x_1, x_2) bundle, $\hat{t} = 0$, then $\hat{t} dz = 0^4$. Inserting these two conditions in $\frac{dx_1}{dp_1}$ in Table 3.1 indicates

$$\text{that } \left. \frac{dx_1}{dp_1} \right|_{\substack{u \text{ constant} \\ \hat{t} = 0}} = \lambda \frac{D_{11}}{D}.$$

Thus $\lambda \frac{D_{11}}{D}$ is a substitution effect in two senses.

Commodities have been switched (substituted) in the face of a price change to hold utility constant and residence sites have been switched (substituted) to hold the total transportation cost for the consumer constant, or in general $\hat{t} = 0$.

Note that $\frac{dx_1}{dy}$, the income effect,

enters $\frac{dx_1}{dp_1}$ in the same manner as in aspatial consumer theory.

Hence the response (measured as dx_1) of a consumer to a change in price of commodity 1 is composed of a generalized substitution effect and an income effect.

$$\frac{dx_1}{dp_1} = \left. \frac{\partial x_1}{\partial p_1} \right|_{\substack{u \text{ constant} \\ \hat{t} = 0}} - x_1 \left. \frac{\partial x_1}{\partial y} \right|_{\substack{\text{prices constant;} \\ \text{sites of markets} \\ \text{fixed; transport} \\ \text{tation rates} \\ \text{constant}}}$$

When $\hat{t} < 0$ then the substitution effect is negative, the orthodox sign. When $u_{12} < 0$ and $\hat{t} < 0$, the income effect is positive. We cannot say whether or under what general conditions the "law of demand" holds, that is $\frac{dx_1}{dp_1} < 0$.

The reason is because a consumer can mitigate the adverse effect of a price rise by shifting his residence site. This he could of course not do in the familiar aspatial model of consumer choice and thus it has been possible in an aspatial context to define general conditions when the "law of demand" holds.

Figure 3.1 illustrates the effect of an increase in the mill price of commodity and on the consumer's equilibrium.

The income effect, bc, occurs when the budget constraint shifts toward the origin and remains parallel to the original constraint. That ratio $\frac{\hat{p}_1}{p_2}$ remains unchanged in the income effect. The substitution effect, ab, occurs when sites are substituted (that is $\frac{\hat{p}_1}{p_2}$ changes until $\hat{t} = 0$) and commodities are substituted (the ratio $\frac{x_1}{x_2}$ changes

while u remains constant) until $\hat{t} = 0$. Observe that we cannot

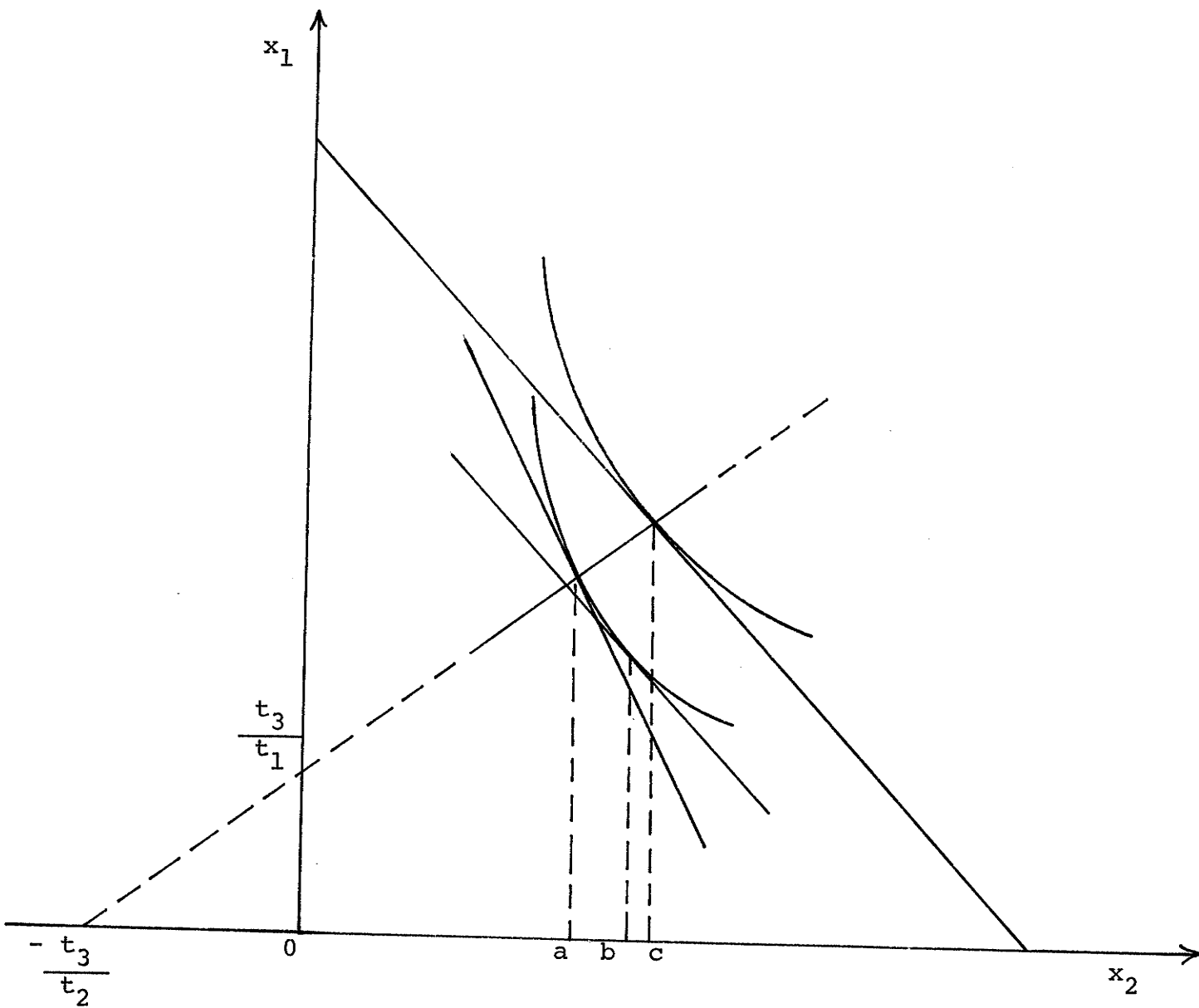


Figure 3.1

tell from the price ratio lines in Figure 3.1 whether we are examining a rise in the mill price of commodity 1 or 2. The income and substitution effects nonetheless are different for the different price changes.

Observe that the effect of shifting the site of market 1, $\frac{dx_1}{dk_1} = -t_1 \frac{dx_1}{dp_1}$, being composed of $\frac{dx_1}{dp_1}$ weighted by $-t_1$ has the income and substitution effects operative.

However, the effect $\frac{dx_1}{dt_1} = (z-k_1) \frac{dx_1}{dp_1} + \lambda x_1 \frac{D_{41}}{D}$ has inherent not only $\frac{dx_1}{dp_1}$ but an additional term $\lambda x_1 \frac{D_{41}}{D}$.

Thus in addition to their being substitution and income effects involved in shifting the transportation cost per unit distance for commodity 1, there is what we shall call the transportation cost effect ($\lambda x_1 \frac{D_{41}}{D}$ for the case of commodity 1). This name derives from the fact that only when either of transportation rates t_1 , t_2 , or t_3 , are perturbed does the effect appear in addition to some simple multiple of the price effect. For other perturbations, total transportation costs (also \hat{t}) are only affected by the consumer's response in shifting x_1 , x_2 , λ and z . However, when transportation cost effects occur we observe those changes arising from a "tilting" in the line $\hat{t} = 0$ in Figure 3.1 in turn arising from a change in either t_1 , t_2 or t_3 .

Note that various cross-effects such as the effect of change in parameter α_i on control variable n_j have a similar form to those effects just analyzed. Also the effects on the location variable z are simple combinations of the price effects on location. Moreover, we see that it was not possible to determine signs of the location response, dz , unequivocally for any parameter shift.

Footnotes:

1. We shall deal throughout this analysis with only two commodities rather than $n > 2$ in order to avoid cluttered pages. The reader can develop the n commodity case with the aid of a standard text. General integrability theorems in aspatial theory require $n > 2$ commodities but we shall not deal with integrability in this paper. Note that u_i is an abbreviation for $\frac{\partial u}{\partial x_i}$ and u_{ij} stands for $\frac{\partial^2 u}{\partial x_i \partial x_j}$. See Henderson and Quandt (1958) for a textbook treatment of the aspatial analysis. We shall use their notation.
2. Consider a non-interior solution. Let t_1 be relatively small and hence line $-\frac{t_3}{t_2}, \frac{t_3}{t_1}$ relatively steep in Figure 22. For certain relatively small incomes, y 's, the tangency of the indifference curve to the income constraint cannot obtain at the point where line $-\frac{t_3}{t_2}, \frac{t_3}{t_1}$ cuts the indifference curve. The consumer would choose an equilibrium for which he was located at either k_1 or k_2 and $t \neq 0$.
3. To obtain this result, recall that we substitute the column headed by dp_1 for dx_1 and solve the new 4 by 4 determinant. We then divide this result by D , the determinant of the original 4 by 4 matrix on the LHS of Table 3.1.
4. This new value of z must fall between k_1 and k_3 in order to keep the analysis simple. We want to restrict our attention to smooth perturbations in the neighbourhood of the original equilibrium. Recall the comments in Footnote 3.

References

Henderson, James M. and Richard E. Quandt, 1958, Microeconomic Theory, New York, McGraw-Hill.

Solow, R.M., 1972, "Congestion, Density and the Use of Land in Transportation", Swedish Journal of Economics (forthcoming).

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